Leonhard Euler
300 years on
Robin Wilson

Read Euler, read Euler, he is the master of us all (Laplace)
Some of Euler’s interests

Theory of numbers
Geometry of a triangle
Musical harmony
Infinite series
Logarithms
Calculus
Mechanics
Complex numbers
Optics
Astronomy
Motion of the moon
Wave motion
Stability of sailing ships . . .
Summary of Euler’s life

1707: Born in Basel (15 April)
1721: University of Basel

1727: to St Petersburg Academy
1733: Chair of Mathematics

1741: to Berlin Academy

1766: returned to St Petersburg
1783: died in St Petersburg
Basel, Switzerland
The Bernoulli family

- Nicolaus (I) 1687 – 1759
  - Nicolaus (II) 1695 – 1726
    - Johann (III) 1744 – 1807
      - Daniel (II) 1751 – 1834
    - Daniel 1700 – 1782
  - Johann (II) 1710 – 1746
    - Jacob (II) 1759 – 1789

- Jacob 1654 – 1705
  - Nicolaus 1662 – 1716
    - Johann 1667 – 1748
Basel, Switzerland
St Petersburg
The 1730s in St Petersburg

1732: $2^{32} + 1$ is divisible by 641

1735: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots = \frac{\pi^2}{6}$

1735: Königsberg bridges

1736: Mechanica

1737: $e$ is irrational
Fermat’s Conjecture

\[ F_n = 2^{2^n} + 1 \] is prime for all \( n \)

\[
\begin{align*}
F_0 &= 2^1 + 1 = 3 \quad \text{– yes} \\
F_1 &= 2^2 + 1 = 5 \quad \text{– yes} \\
F_2 &= 2^4 + 1 = 17 \quad \text{– yes} \\
F_3 &= 2^8 + 1 = 257 \quad \text{– yes} \\
F_4 &= 2^{16} + 1 = 65537 \quad \text{– yes}
\end{align*}
\]

\[ \text{Is } F_5 = 2^{32} + 1 = 4294967297 \text{ prime?} \]
Euler: $F_5 = 2^{32} + 1$ is divisible by 641

Proof

Note that $641 = 2^4 + 5^4 = (2^7 \times 5)^4 + 1$.

So $2^{32} + 1 = 2^{28} \times (2^4 + 5^4) - (2^7 \times 5)^4 + 1$

$= (2^{28} \times 641) - (641 - 1)^4 + 1 = 641 \times K$,

so 641 divides $2^{32} + 1$. 
Euler’s arithmetic

François Arago: *Euler calculated without any apparent effort, just as men breathe, as eagles sustain themselves in the air.*

Find numbers $a$, $b$, $c$, $d$ such that $a + b$, $a + c$, $a + d$, $b + c$, $b + d$, $c + d$ are all perfect squares
Euler’s arithmetic

Find numbers \(a, b, c, d\) such that \(a + b, a + c, a + d, b + c, b + d, c + d\) are all perfect squares

Euler’s solution:

\[
\begin{align*}
a &= 18530 & b &= 38114 \\
c &= 45986 & d &= 65570
\end{align*}
\]

\([56644, 64516, 84100, 84100, 103684, 111556]\)
Some infinite series

\[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 2\]

\[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = ?\]

\[\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \ldots = ?\]

\[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = ?\]
Euler’s constant $\gamma$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \ldots = ?$$

If $S = (1 + \frac{1}{2}) + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6}) + \ldots$, then $S > (\frac{1}{2} + \frac{1}{2}) + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{6} + \frac{1}{6}) + \ldots = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = S$

So $S > S$, contradiction: the series has no sum.

Euler

If $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} - \log n$
then $f(n) \to$ a constant $\gamma \ (\approx 0.5772\ldots)$
The ‘Basel problem’

In 1735 Euler proved:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots = \frac{\pi^2}{6}$$
Euler’s zeta-function

Euler proved:

$$\zeta(k) = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots = \frac{\pi^2}{6}$$

$$\zeta(2) = \frac{\pi^2}{6}$$

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(6) = \frac{\pi^6}{945}$$

… up to

$$\zeta(26) = \frac{1315862\pi^{26}}{110944815976030578125}$$

$$\zeta(1)$$ doesn’t exist
Königsberg bridges problem (1735)

Can you cross each of the seven bridges exactly once?
Euler and the Königsberg bridges

This question is so banal, but seemed to me worthy of attention in that geometry, nor algebra, nor even the art of counting was sufficient to solve it.

In view of this, it occurred to me to wonder whether it belonged to the geometry of position, which Leibniz had once so much longed for.

And so, after some deliberation, I obtained a simple, yet completely established, rule with whose help one can immediately decide for all examples of this kind, with any number of bridges in any arrangement, whether such a round trip is possible, or not...
Solving the Königsberg bridges problem
Euler’s Königsberg solution

If the number of bridges is even for all areas, then the journey is possible, starting from any area.

If the number of bridges is odd for exactly two areas, then the journey is possible, starting in one area and ending in the other.

If the number of bridges is odd for more than two areas, then such a journey is impossible

So the Königsberg bridges problem has no solution
The modern approach (using graph theory)

Can you draw this picture in one continuous stroke?

– NOT drawn by Euler
Euler’s mechanics

1736: *Mechanica*, on the dynamics of a particle

1750: Motion of a rigid body: Euler’s equations of motion
Moments of inertia

1776: Theorem on the rotation of a body about a point

Much of this work involved differential equations
Berlin Academy of Sciences
Euler & Frederick the Great
1741–1766 in Berlin

1744: Calculus of variations
1748: *Introductio in Analysin Infinitorum*
  \[ e^{ix} = \cos x + i \sin x \]
  Functions
  Conics & quadrics
  Partitions
1749: Theory of tides
  Motion of the moon
1749/50: Vibrating strings
  Differential equations
  Waves
1750: Polyhedron formula
1755: *Calculi Differentialis*
1759: Knight’s tour problem
1760: Differential geometry
Calculus of variations (1741)

Euler: ‘Since the fabric of the Universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.’
Euler’s Introductio in Analysin Infinitorum (1748)
Introduction – the number e

\[ e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \] for large \( n \)

\[ e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \]
Euler’s ‘famous formula’

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \]

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \]

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \]

So: \[ e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots \]

\[ = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right) \]

\[ + i \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \right) \]

\[ e^{ix} = \cos x + i \sin x \]

\[ e^{i\pi} = -1, \text{ so } e^{i\pi} + 1 = 0 \]
The Derangement Problem

Given any $n$ letters $a, b, c, d, e, \ldots$, in how many ways can they be arranged so that none is in its original position?

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
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<tbody>
<tr>
<td>$D_n$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>9</td>
<td>44</td>
<td>265</td>
<td>1854</td>
<td>14833</td>
<td>...</td>
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Euler: \[ D_n = (n-1)D_{n-2} + (n-1)D_{n-1} \]
\[ D_n = nD_{n-1} + (-1)^n \]
\[ \Rightarrow D_n = n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!} \right\} \]
\[ \approx n!/e \]

$n = 8$, $D_8 = 14833$, $n!/e = 14832.9$
Introdução – Conics and Quadrics

Equation: \( yy = \alpha + \beta x + \gamma xx \)

or \( y^2 = \alpha + \beta x + \gamma x^2 \)

\( \gamma \) negative:
- ellipse

\( \gamma = 0 \):
- parabola

\( \gamma \) positive
- hyperbola
Partitions of numbers
Leibniz: ‘divulsions of integers’

Split a number into smaller ones
1 = 1 (1 way)  
2 = 2 or 1 + 1 (2 ways)  
3 = 3 or 2 + 1 or 1 + 1 + 1 (3 ways)  
4 = 4 or 3+1 or 2+2 or 2+1+1 or 1+1+1+1 (5 ways)  
5 = 5 or 4+1 or 3+2 or 3+1+1 or 2+2+1 or ... or ... (7 ways)  
   ...  

p(1) = 1, p(2) = 2, p(3) = 3, p(4) = 5, p(5) = 7, p(6) = 11,  
p(10) = 42, p(20) = 627, p(30) = 5604, p(40) = 37338, ... ,  
p(200) = 3,972,999,029,388
Euler’s Pentagonal Number Theorem

Look at the ‘washing line’:

\[ F(x) = 1 + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \ldots \]

\[ = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + \ldots \]

In the *Introductio* Euler proved that

\[ F(x) = (1 - x)^{-1} \times (1 - x^2)^{-1} \times (1 - x^3)^{-1} \times (1 - x^4)^{-1} \times \ldots \]

\[ = 1 / \{(1 - x)(1 - x^2)(1 - x^3)(1 - x^4)\ldots\} \]

and later that

\[ (1 - x) \times (1 - x^2) \times (1 - x^3) \times (1 - x^4) \times \ldots \]

\[ = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots \]

with exponents \( k(3k \pm 1)/2 \), the ‘pentagonal numbers’.
Euler’s Partition Formula

Multiplying these expressions together we get:
\[
\{1 + p(1)x + p(2)x^2 + p(3)x^3 + p(4)x^4 + \ldots \} \\
\times \{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots \} = 1.
\]

Isolating the term in \(x^n\) and rearranging the result, we get:
\[
p(n) = p(n - 1) + p(n - 2) - p(n - 5) - p(n - 7) \\
+ p(n - 12) + p(n - 15) - \ldots
\]

from which each successive partition number \(p(n)\) can be calculated from the previous ones.

So \(p(11) = p(10) + p(9) - p(6) - p(4) = 42 + 30 - 11 - 5 = 56.\)

Euler went up to \(p(65)\)
Partition numbers up to \(p(200)\)

(calculated by Percy MacMahon)
Euler's polyhedron formula: \( F + V = E + 2 \)

- **cube**
  - 6 faces, 8 vertices, 12 edges
  - and \( 6 + 8 = 12 + 2 \)

- **dodecahedron**
  - 12 faces, 20 vertices, 30 edges
  - and \( 12 + 20 = 30 + 2 \)

- **great rhombicosidodecahedron**
  - 62 faces, 120 vertices, 180 edges
  - and \( 62 + 120 = 180 + 2 \)
Euler’s letter to C. Goldbach (1750)

6. Interdum solidus solidi plani includit, propositionem ex numero hanc hanc et numeri angularium solidorum binarum sequentiam numerum aciem.

Lei est $N + S = A + 2$, seu $N + S = \frac{1}{2}A + 2 = \frac{1}{3}P + 2$.

7. Impossibile est ut sit $A \geq 6$ et $N + S$ vel $A + 6 + S$.

8. Impossibile est ut sit $N + S$ vel $S + A + 6 + 2$.

9. Nihilum formarum solidarum aucupat omnes hanc, sed 5 planorum

10. Summa omnium angularium planorum, que in amphibio solidi expressa 

11. Summa omnium angularium planorum, quatuor ad quatuor est angulo 

Exemplo fit prima triangulare sit $a$.

1. numerus hæc $H = a$

2. numerus ang. $S = b$

3. numerus actuum $(ab, ac, bc, ad, bd, ef, de, df, ef)$ $A = 9$
The Eulerian calculus was not about curves, but about functions: formal expressions that can be differentiated and integrated. The emphasis is on mathematics as a science of formal expressions: a restructured mathematical theory.

A function of a variable is any analytical expression containing that variable and numbers or constants.
Euler’s Integral Calculus (1768–70)
Knight’s-tour Problem (1759)
1766–1783 in St Petersburg

1767: Euler line of a triangle
1768/74: Letters to a German Princess
1768–70: *Calculi Integralis* (3 volumes)
1770: Algebra / number theory
1771: *Dioptrica* (optics)
1773: Sailing of ships
1774: Astronomy book
1776: Motion of rigid bodies
1776: 775-page treatise on the motion of the moon
1782: Magic and Latin squares / 36 Officers problem
1783: Died 7/18 September
The Euler line of a triangle (1767)

O, C and M are collinear

$OC = 2CM$
Letters to a German Princess
200 letters from 1760–62 to the Princess of Anhalt-Dessau on elementary science

astronomy
light
sound
gravity
magnetism

. . .
Euler’s number theory

Perfect numbers
(such as 6, 28 and 496)

Euler classified all even perfect numbers:

\[ 2^{n-1}(2^n - 1), \]
with \( 2^n - 1 \) prime

Fermat’s Last Theorem

If \( n > 2 \), then \( x^n + y^n \neq z^n \)

Euler (1770s): true for \( n = 3, 4 \)

Fermat’s Little Theorem

\[ a^{p-1} - 1 \text{ is divisible by } p \]
(if \( a \) isn’t

Example: \( a = 48, p = 73: \)

\[ 48^{72} - 1 \text{ is divisible by } 73 \]

Euler’s generalization for \( n \)

(Euler \( \phi \)-function):

\[ a^{\phi(n)} = 1 \text{ is divisible by } n \]
Three Latin squares (3 × 3, 4 × 4, 5 × 5)

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7 × 7 latin squares
Sudoku puzzles
(1 to 9 in each row, column and block)

These are not due to Euler
Sixteen court card puzzle (1612)

The values (J, Q, K, A) form a latin square – and so do the suits

\[\begin{array}{cccc}
K\spadesuit & Q\diamondsuit & J\spadesuit & A\heartsuit \\
J\heartsuit & A\spadesuit & K\diamondsuit & Q\clubsuit \\
A\diamondsuit & J\clubsuit & Q\heartsuit & K\spadesuit \\
Q\spadesuit & K\heartsuit & A\clubsuit & J\spadesuit
\end{array}\]

‘Orthogonal 4 × 4 latin squares’
Orthogonal 5 × 5 latin squares

Each chess-piece and colour appear together just once
Each capital and small letter appear together just once
Orthogonal $6 \times 6$ latin squares?

Euler’s 36 Officers Problem (1782)

Arrange 36 officers, one of each of six ranks and one of each of six regiments, in a $6 \times 6$ square array, so that each row and each column contains exactly one officer of each rank and exactly one of each regiment.

1. Une question fort curieuse, qui a exercé pendant quelque temps la sagacité de bien du monde, m’a engagé à faire les recherches suivantes, qui semblent ouvrir une nouvelle carrière dans l’Analyse et en particulier dans la doctrine des combinaisons. Cette question rouloit sur une assemblée de 36 officiers, de six différens grades et tirés de six régimens différens, qu’il s’agissoit de ranger dans un quarré de manière que sur chaque ligne, tant horizontale que verticale, il se trouvat six officiers tant de différens caracteres que de régimens différens. Or, après toutes les peines qu’on s’est donées pour résoudre ce problème, on a été obligé de reconnitre qu’un tel arrangement est absolument impossible, quoiqu’on ne puisse pas en donner de démonstration rigoureuse.
Euler’s Conjecture

Observing that one can easily construct orthogonal Latin squares of sizes
$3 \times 3$, $4 \times 4$, $5 \times 5$ and $7 \times 7$,
and unable to solve the 36 Officers Problem, Euler conjectured:

Constructing orthogonal $n \times n$ Latin squares is impossible when

$n = 6, 10, 14, 18, 22, \ldots$,

but can be done in all other cases.
Euler was wrong!

In 1958–60, R. C. Bose, S. Shrikhande and E. T. Parker showed that orthogonal latin squares exist for all of these values of $n$, except for $n = 6$. 
Orthogonal $10 \times 10$ latin squares
On the 7th of September 1783, after amusing himself with calculating on a slate the laws of the ascending motion of air balloons, the recent discovery of which was then making a noise all over Europe, he dined with Mr Lexell and his family, talked of Herschel’s planet (Uranus), and of the calculations which determine its orbit.

A little after, he called his grandchild, and fell a playing with him as he drank tea, when suddenly the pipe, which he held in his hand, dropped from it, and he ceased to calculate and to breathe.